

# Reliable equation of state for composite bosons in the 2D BCS-BEC crossover

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**Abstract** We briefly discuss recent experiments on the BCS-BEC crossover with ultracold alkali-metal atoms both in three-dimensional configurations and two-dimensional ones. Then we analyze the quantum-field-theory formalism used to describe an attractive  $D$ -dimensional Fermi gas taking into account Gaussian fluctuations. Finally, we apply this formalism to obtain a reliable equation of state of the 2D system at low temperatures in the BEC regime of the crossover by performing a meaningful dimensional regularization of the divergent zero-point energy of collective bosonic excitations.

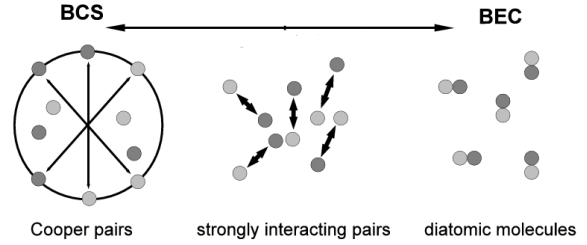
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## 1 BCS-BEC crossover with ultracold atoms

In 2004 the 3D BCS-BEC crossover has been observed with ultracold gases made of fermionic  $^{40}\text{K}$  and  $^6\text{Li}$  alkali-metal atoms [1, 2, 3, 4]. As schematically shown in Fig. 1, this crossover is obtained by changing with a Feshbach resonance the s-wave scattering length  $a_F$  of the inter-atomic potential. There are three characteristic regimes which depend on the value of the scattering length  $a_F$  [5]:

- $a_F \rightarrow 0^-$ , that is the BCS regime of weakly-interacting Cooper pairs;
- $a_F \rightarrow \pm\infty$ , that is unitarity limit of strongly-interacting Cooper pairs;
- $a_F \rightarrow 0^+$ , that is the BEC regime of bosonic dimers.

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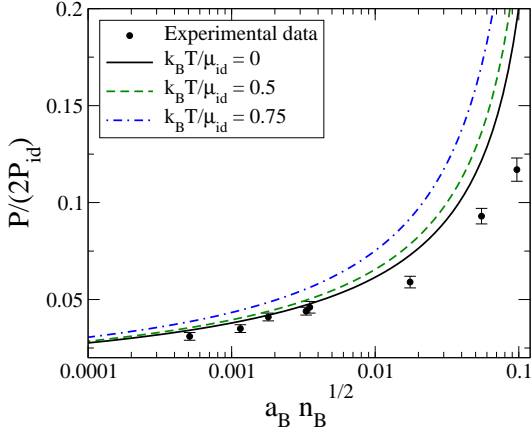
**Fig. 1** The figure shows the evolution from the BCS limit with large, spatially overlapping Cooper pairs to the BEC limit with tightly bound molecules. The system is a fermionic gas made of attractive two-spin-component atoms with s-wave scattering length  $a_F$ . Adapted from Ref. [5].

The crossover from a BCS superfluid ( $a_F < 0$ ) to a BEC of molecular pairs ( $a_F > 0$ ) has been investigated experimentally around a Feshbach resonance, where the s-wave scattering length  $a$  diverges ( $a_F = \pm\infty$ ), and it has been shown that the system is metastable [1, 2, 3, 4]. The detection of quantized vortices under rotation [6] has clarified that this dilute gas of ultracold atoms is superfluid. Usually the BCS-BEC crossover is analyzed in terms of

$$y = \frac{1}{k_F a_F} \quad (1)$$

the inverse scaled interaction strength, where  $k_F = (3\pi^2 n)^{1/3}$  is the Fermi wave number and  $n$  the total fermionic density. The system is dilute because  $r_e k_F \ll 1$ , with  $r_e$  the effective range of the inter-atomic potential.

In 2014 also the 2D BCS-BEC crossover has been achieved [7] with a quasi-2D Fermi gas of  $^6\text{Li}$  atoms with widely tunable s-wave interaction, measuring the pressure  $P$  vs the gas parameter  $a_B n_B^{1/2}$ , with  $a_B = a_F / (2^{1/2} e^{1/4})$  the bosonic scattering length between



**Fig. 2** Scaled pressure  $P/(2P_{id})$  of the 2D Bose gas as a function of the gas parameter  $a_B n_B^{1/2}$ , where  $P_{id}$  is the pressure of an ideal 2D gas,  $a_B$  is the s-wave scattering length of bosons, and  $n_B$  is the bosonic 2D density. The filled circles with error bars are the experimental data [7]. The curves are obtained with our beyond-mean-field theory (see below and [8]).

molecules (see below and [8]) and  $n_B = n/2$  the bosonic density. In Fig. 2 we plot the pressure  $P$  of the system as a function of the gas parameter.

Fig. 2 shows a good agreement between the experimental data [7] and our theoretical curves only in the deep weak-coupling regime  $a_B n^{1/2} < 0.01$  and assuming a very small scaled temperature  $k_B T / \mu_{id}$ . In the next two sections we shall discuss some details of our beyond mean-field theory [8,9].

## 2 Theory for a $D$ -dimensional Fermi superfluid

To study the attractive  $D$ -dimensional Fermi liquid we adopt the path integral formalism [10]. The partition function  $\mathcal{Z}$  of the uniform system with fermionic fields  $\psi_s(\mathbf{r}, \tau)$  at temperature  $T$ , in a  $D$ -dimensional volume  $L^D$ , and with chemical potential  $\mu$  reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{1}{\hbar} S \right\}, \quad (2)$$

where  $(\beta \equiv 1/(k_B T))$  with  $k_B$  Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D \mathbf{r} \mathcal{L} \quad (3)$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \quad (4)$$

where  $g$  is the attractive strength ( $g < 0$ ) of the s-wave coupling.

Through the usual Hubbard-Stratonovich transformation the Lagrangian density  $\mathcal{L}$ , quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field  $\Delta(\mathbf{r}, \tau)$  so that:

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\Delta, \bar{\Delta}] \exp \left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})}{\hbar} \right\}, \quad (5)$$

where

$$S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D \mathbf{r} \mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) \quad (6)$$

and the (exact) effective Euclidean Lagrangian density  $\mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})$  reads

$$\begin{aligned} \mathcal{L}_e = & \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s \\ & + \bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}. \end{aligned} \quad (7)$$

We want to investigate the effect of fluctuations of the gap field  $\Delta(\mathbf{r}, t)$  around its mean-field value  $\Delta_0$  which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = \Delta_0 + \eta(\mathbf{r}, \tau), \quad (8)$$

where  $\eta(\mathbf{r}, \tau)$  is the complex field which describes pairing fluctuations.

In particular, we are interested in the grand potential  $\Omega$ , given by

$$\Omega = -\frac{1}{\beta} \ln(\mathcal{Z}) \simeq -\frac{1}{\beta} \ln(\mathcal{Z}_{mf} \mathcal{Z}_g) = \Omega_{mf} + \Omega_g, \quad (9)$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\} \quad (10)$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\eta, \bar{\eta}] \exp \left\{ -\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\} \quad (11)$$

is the partition function of Gaussian pairing fluctuations.

To make a long story short, one finds that in the gas of paired fermions there are two kinds of elementary excitations [10,11,12]: fermionic single-particle excitations with energy

$$E_{sp}(k) = \sqrt{\left( \frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta_0^2}, \quad (12)$$

where  $\Delta_0$  is the pairing gap, and bosonic collective excitations with energy

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m} \left( \lambda \frac{\hbar^2 q^2}{2m} + 2 m c_s^2 \right)}, \quad (13)$$

where  $\lambda$  is the first correction to the familiar low-momentum phonon dispersion  $E_{col}(q) \simeq c_s \hbar q$  and  $c_s$

is the sound velocity. Notice that both  $\lambda$  and  $c_s$  depend on the chemical potential  $\mu$  [12].

Moreover, at the Gaussian level, the total grand potential reads [10,12]

$$\Omega = \Omega_{mf} + \Omega_g, \quad (14)$$

where

$$\Omega_{mf} = -\frac{\Delta_0^2}{g} L^D + \Omega_F^{(0)} + \Omega_F^{(T)} \quad (15)$$

is the mean-field grand potential with

$$\Omega_F^{(0)} = -\sum_{\mathbf{k}} \left( E_{sp}(k) - \frac{\hbar^2 k^2}{2m} + \mu \right) \quad (16)$$

the zero-point energy of fermionic single-particle excitations,

$$\Omega_F^{(T)} = \frac{2}{\beta} \sum_{\mathbf{k}} \ln(1 + e^{-\beta E_{sp}(k)}) \quad (17)$$

the finite-temperature grand potential of the fermionic single-particle excitations.

The grand-potential of Gaussian fluctuations reads

$$\Omega_g = \Omega_{g,B}^{(0)} + \Omega_{g,B}^{(T)}, \quad (18)$$

where

$$\Omega_{g,B}^{(0)} = \frac{1}{2} \sum_{\mathbf{q}} E_{col}(q) \quad (19)$$

is the zero-point energy of bosonic collective excitations and

$$\Omega_{g,B}^{(T)} = \frac{1}{\beta} \sum_{\mathbf{q}} \ln(1 - e^{-\beta E_{col}(q)}) \quad (20)$$

is the finite-temperature grand potential of the bosonic collective excitations.

Both  $\Omega_F^{(0)}$  and  $\Omega_{g,B}^{(0)}$  are ultraviolet divergent in any dimension  $D$  ( $D = 1, 2, 3$ ) and the regularization of these divergent terms is complicated by the fact that one also must take into account the BCS-BEC crossover [12,8].

### 3 Results of the two-dimensional Fermi superfluid

In the analysis of the two-dimensional attractive Fermi gas one must remember that, contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular, the binding energy  $\epsilon_b > 0$  of two fermions can be written in terms of the positive 2D fermionic scattering length  $a_F$  as

$$\epsilon_b = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{ma_F^2}, \quad (21)$$

where  $\gamma = 0.577\dots$  is the Euler-Mascheroni constant [13]. Moreover, the attractive (negative) interaction

strength  $g$  of s-wave pairing is related to the binding energy  $\epsilon_b > 0$  of a fermion pair in vacuum by the expression [14]

$$-\frac{1}{g} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \frac{1}{2}\epsilon_b}. \quad (22)$$

In the 2D BCS-BEC crossover, at zero temperature ( $T = 0$ ) the mean-field grand potential  $\Omega_{mf}$  can be written as [12,14]

$$\Omega_{mf} = -\frac{mL^2}{2\pi\hbar^2} \left( \mu + \frac{1}{2}\epsilon_b \right)^2 \quad (23)$$

with  $\epsilon_b > 0$ . Using

$$n = -\frac{1}{L^2} \frac{\partial \Omega_{mf}}{\partial \mu} \quad (24)$$

one immediately finds the chemical potential  $\mu$  as a function of the number density  $n = N/L^2$ , i.e.

$$\mu = \frac{\pi\hbar^2}{m} n - \frac{1}{2}\epsilon_b. \quad (25)$$

In the BCS regime, where  $\epsilon_b \ll \epsilon_F$  with  $\epsilon_F = \pi\hbar^2 n/m$ , one finds  $\mu \simeq \epsilon_F > 0$  while in the BEC regime, where  $\epsilon_b \gg \epsilon_F$  one has  $\mu \simeq -\epsilon_b/2 < 0$ .

Performing dimensional regularization of Gaussian fluctuations, we have recently found [8] that the zero-temperature total grand potential is

$$\Omega = \Omega_{mf} + \Omega_g = -\frac{mL^2}{\pi\hbar^2} \left( \mu + \frac{1}{2}\epsilon_b \right)^2 \ln \left( \frac{\epsilon_b}{2(\mu + \frac{1}{2}\epsilon_b)} \right). \quad (26)$$

in the deep BEC regime. Introducing  $\mu_B = 2(\mu + \epsilon_b/2)$  as the chemical potential of composite bosons with mass  $m_B = 2m$  and density  $n_B = n/2$ , the zero-temperature total grand potential can be rewritten as

$$\Omega = -\frac{m_B L^2}{8\pi\hbar^2} \mu_B^2 \ln \left( \frac{\epsilon_0}{\mu_B} \right), \quad (27)$$

that is exactly the Popov equation of state of 2D weakly-interacting bosons [15] provided that we identify the parameter

$$\epsilon_0 = \frac{4}{e^{2\gamma+1/2}} \frac{\hbar^2}{m_B a_B^2} \quad (28)$$

of the Popov theory of bosons with scattering length  $a_B$  [16] with the binding energy

$$\epsilon_b = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{ma_F^2} \quad (29)$$

of paired fermions with scattering length  $a_F$  [13]. Thus, we find [8]

$$\boxed{a_B = \frac{1}{2^{1/2}e^{1/4}} a_F}. \quad (30)$$

The value  $a_B/a_F = 1/(2^{1/2}e^{1/4}) \simeq 0.551$  is in full agreement with other theoretical predictions:  $a_B/a_F = 0.56$  obtained from four-body scattering theory [17],

$a_B/a_F = 0.55(4)$  obtained by Monte Carlo calculations [18], and  $a_B/a_F = 0.56$  obtained very recently by using Gaussian fluctuations with convergence-factor regularization [19].

At finite temperature ( $T \neq 0$ ) the pressure  $P$  is immediately obtained using the thermodynamic formula  $P = -\Omega/L^2$ . Taking into account that the main thermal contribution is due to collective bosonic excitations, we obtain [9] from Eqs. (20) and (27) the finite-temperature pressure

$$P = \frac{m_B}{8\pi\hbar^2} \mu_B^2 \left[ \ln \left( \frac{\epsilon_0}{\mu_B} \right) + 4\zeta(3) \left( \frac{k_B T}{\mu_B} \right)^3 \right], \quad (31)$$

and also, by using  $n_B = \left( \frac{\partial \Omega}{\partial \mu_B} \right)_{T, L^2}$ , the bosonic density

$$n_B = \frac{m_B}{4\pi\hbar^2} \mu_B \left[ \ln \left( \frac{\epsilon_0}{\mu_B e^{1/2}} \right) - 2\zeta(3) \left( \frac{k_B T}{\mu_B} \right)^3 \right] \quad (32)$$

where  $\zeta(x)$  is the Riemann zeta function and  $\zeta(3) = 1.20205$ . Eqs. (31) and (32) give, at fixed  $k_B T/\mu_B$ , a parametric formula for the the pressure  $P$  as a function of the density  $n_B$  where  $\mu_B$  is the dummy parameter (see Fig. 2). Thus, we have a reliable equation of state for composite bosons in the 2D BEC-BEC crossover at low temperatures, i.e. when the system is well below the Berezinsky-Kosterlitz-Thouless critical temperature of the superfluid-normal transition [10].

## 4 Conclusions

We have shown that the  $D$ -dimensional superfluid Fermi gas in the BCS-BEC crossover has a divergent zero-point energy due to fermionic single-particle excitations (mean-field) and bosonic collective excitations (Gaussian fluctuations). However, the regularization of the divergent zero-point energy gives remarkable analytical results for composite bosons in two dimensions [8]: a reliable 2D equation of state and an analytical formula connecting the scattering length  $a_B$  between composite bosons and the scattering  $a_F$  between fermionic atoms. Finally, we notice that also in three-dimensions one can regularize the divergent zero-point energy due to fermionic and bosonic excitations [20, 21, 22]. In particular, by performing a cutoff regularization and renormalization of Gaussian fluctuations, we have found very recently [23] that  $a_B = (2/3)a_F$  for composite bosons in the 3D BCS-BEC crossover.

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